IE 361 Module 45

Design and Analysis of Experiments Part 5 (Confidence Intervals for Two-Way Analyses)

Reading: Section 5.2 *Statistical Methods for Quality Assurance*

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Before basing serious real world decisions on the perceived sizes of effects of experimental factors on a response, it is important to have some comfort that one is seeing more than is explainable as "just experimental variation." One must be reasonably certain that the effects one sees are more than just background noise. In the present two-way factorial context, that means that beyond computing fitted main effects and interactions a_i, b_j, and ab_{ij} one really needs some way of judging whether they are clearly "more than just noise." Attaching margins of error to them is a way of doing this, and the key to seeing how to find relevant margins of error is to recognize that fitted effects are linear combinations of $\bar{y}$'s, $\hat{L}$'s in the notation of Module 42.

Take, for example, the quantity b_2 in any 2×3 study (like the glass-phosphor study). This is

$$\begin{aligned} b_2 &= \bar{y}_{.2} - \bar{y}_{..} \\ &= \frac{1}{2}\left(\bar{y}_{12} + \bar{y}_{22}\right) \\ &\quad -\frac{1}{6}\left(\bar{y}_{11} + \bar{y}_{12} + \bar{y}_{13} + \bar{y}_{21} + \bar{y}_{22} + \bar{y}_{23}\right) \\ &= -\frac{1}{6}\bar{y}_{11} + \frac{1}{3}\bar{y}_{12} - \frac{1}{6}\bar{y}_{13} - \frac{1}{6}\bar{y}_{21} + \frac{1}{3}\bar{y}_{22} - \frac{1}{6}\bar{y}_{23} \end{aligned}$$

which is indeed of the form $\hat{L} = c_1\bar{y}_1 + c_2\bar{y}_2 + \cdots + c_r\bar{y}_r$.

The fact that fitted factorial effects are $\hat{L}$'s means that there are corresponding linear combinations of population/theoretical means μ (there are corresponding L's) and the methods of Module 42 can be used to make confidence limits based on the fitted effects ... to find margins of error to attach to the fitted effects.

We may use

$$\widehat{Effect} \pm t s_{\text{pooled}} \sqrt{\frac{c_{11}^2}{n_{11}} + \cdots + \frac{c_{IJ}^2}{n_{IJ}}}$$

to make confidence limits based on fitted effects $\widehat{Effect}$ if we can see what are the appropriate sums to put under the square root in the formula. And there are some fairly simple "hand calculation" formulas for what goes under the root in the case of two-way studies, particularly for cases where all $n_{ij} = m$ (there is a common fixed sample size). (Standard jargon is that this is the **balanced data** case.) Table 5.5 of *SMQA* gives the balanced data formulas and is essentially reproduced on panel 5. (Table 5.6 of *SMQA* gives general formulas not requiring balanced data.)

Here's Table 5.5 of *SMQA*.

L	$\hat{L}$	$\frac{c_{11}^2}{n_{11}} + \cdots + \frac{c_{IJ}^2}{n_{IJ}}$
$\alpha\beta_{ij}$	ab_{ij}	$\frac{(I-1)(J-1)}{mIJ}$
α_i	a_i	$\frac{I-1}{mIJ}$
$\alpha_i - \alpha_{i'}$	$a_i - a_{i'}$	$\frac{2}{mJ}$
β_j	b_j	$\frac{J-1}{mIJ}$
$\beta_j - \beta_{j'}$	$b_j - b_{j'}$	$\frac{2}{mI}$

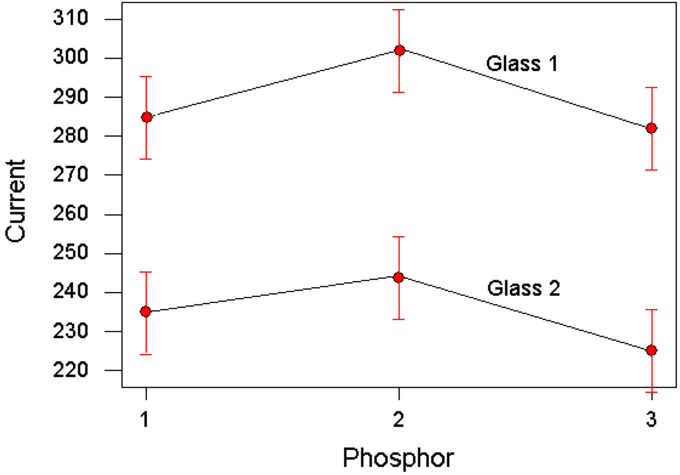
In the glass-phosphor study, $I = 2$, $J = 3$, and the common sample size is $m = 3$. So a margin of error to associate with any of the fitted interactions ab_{ij} based on 95% two-sided confidence limits is

$$
\begin{aligned}
ts_{\text{pooled}}\sqrt{\frac{(2-1)(3-1)}{3(2)(3)}} &= 2.179\,(8.3)\sqrt{\frac{1}{9}} \\
&= 6.0\,\mu\text{A}
\end{aligned}
$$

and reviewing the table giving the fitted interactions, we see that all $I \times J = 6$ of them are smaller than this in absolute value. The lack of parallelism seen on the interaction plots is not only small in comparison to the size of fitted main effects, it is in fact "down in the noise range." This is quantitative confirmation of the story in this regard told on an interaction plot enhanced with error bars repeated from before on the next panel.

Confidence Intervals for Two-Way Factorial Effects

Example 43-1 continued (Interactions)



Confidence Intervals for Two-Way Factorial Effects

Example 43-1 continued (A Main Effects)

A margin of error to be applied to the difference in fitted Factor A main effects ($a_2 - a_1 = (\bar{y}_{2.} - \bar{y}_{..}) - (\bar{y}_{1.} - \bar{y}_{..}) = \bar{y}_{2.} - \bar{y}_{1.} = -54.44$) is then

$$
\begin{aligned}
ts_{\text{pooled}}\sqrt{\frac{2}{3\cdot 3}} &= 2.179\,(8.3)\,\sqrt{\frac{2}{9}} \\
&= 8.6\,\mu\mathrm{A}
\end{aligned}
$$

Since $|-54.44| > 8.6$ there is then a difference between the glass 1 and glass 2 main effects that is clearly more than experimental noise, again providing quantitative confirmation of what seems "obvious" on the plot on panel 7.

And finally, a margin of error to be applied to any difference in fitted Factor B main effects $\left(b_j - b_{j'} = (\bar{y}_{.j} - \bar{y}_{..}) - (\bar{y}_{.j'} - \bar{y}_{..}) = \bar{y}_{.j} - \bar{y}_{.j'}\right)$ is then

$$
\begin{aligned}
ts_{\text{pooled}}\sqrt{\frac{2}{3 \cdot 2}} &= 2.179\,(8.3)\,\sqrt{\frac{1}{3}} \\
&= 10.5\,\mu\mathrm{A}
\end{aligned}
$$

Looking again at the fitted phosphor main effects and their differences, the difference between phosphor 2 main effects and those of either of the other two phosphors is clearly more than experimental noise, but the observed difference between phosphor 1 and 3 main effects is "in the noise range." Once again, this is quantitative confirmation of what in retrospect seems "obvious" on the plot on panel 7.

Perspective

The virtue of inventing numerical measures like the main effects and interactions we have met here and learning how to attach margins of error to them, over what for example can be seen on a plot like that on panel 7, is that the numerical ideas generalize to cases with *many factors*, where there is no obvious way to make pictures to allow us to "see" what is going on. In the next module we begin to consider 3 (and higher) way factorials, placing primary attention on the case where every factor has just 2 levels, and make heavy use of such fitted factorial effects.