IE 361 Module 9

A Simple Method for Separating Components of Variation in Measurements

Reading: Section 2.3.1 *Statistical Methods for Quality Assurance*

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A Relatively Simple First Method for Separating Contributions of Two Sources of Variation

In Module 5 we observed that

1. repeated measurement of a single measurand with a single device allows one to estimate σ_{device}, and

2. single measurements made on multiple measurands from a stable process using a linear device allow one to estimate
$$\sigma_y = \sqrt{\sigma_x^2 + \sigma_{\text{device}}^2}$$

and remarked that these facts might allow one to somehow find a way to estimate σ_x (a process standard deviation) alone. Our first goal in this module is to provide one simple method of doing this.

The next figure illustrates a data collection plan that combines the elements 1. and 2. above.

One Method of Separating Process and Measurement Variation
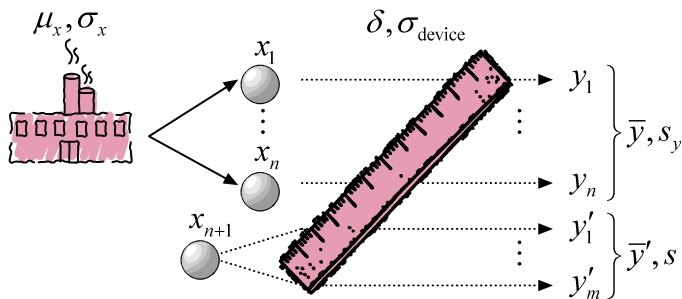
y_i's $\sim$ ind $\left(\mu_x + \delta, \sqrt{\sigma_x^2 + \sigma_{\text{device}}^2}\right)$ independent of

y_i''s $\sim$ ind $\left(x_{n+1} + \delta, \sigma_{\text{device}}\right)$

Figure: One Possible Data Collection Plan for Estimating a Process Standard Deviation, σ_x

One Method of Separating Process and Measurement Variation

Here we will use the notation y for the single measurements on n items from the process and the notation y' for the m repeat measurements on a single measurand. The sample standard deviation of the y's, s_y, is a natural empirical approximation for $\sigma_y = \sqrt{\sigma_x^2 + \sigma_{\text{device}}^2}$ and the sample standard deviation of the y''s, s, is a natural empirical approximation for σ_{device}. That suggests that one estimate the process standard deviation with

$$\widehat{\sigma_x} = \sqrt{\max\left(0, s_y^2 - s^2\right)} \tag{1}$$

as indicated in display (2.3), page 20 of *SQAME*. (The maximum of 0 and $s_y^2 - s^2$ under the root is there simply to ensure that one is not trying to take the square root of a negative number in the rare case that s exceeds s_y.)

One Method of Separating Process and Measurement Variation

$\widehat{\sigma_x}$ is not only a sensible single number estimate of σ_x, but can also be used to make approximate confidence limits for the process standard deviation. The so-called Satterthwaite approximation suggests that one use

$$\widehat{\sigma_x}\sqrt{\frac{\hat{\nu}}{\chi^2_{\text{upper}}}} \text{ and } \widehat{\sigma_x}\sqrt{\frac{\hat{\nu}}{\chi^2_{\text{lower}}}} \text{ as limits for } \sigma_x$$

where appropriate "approximate degrees of freedom" are

$$\hat{\nu} = \frac{\widehat{\sigma_x}^4}{\dfrac{s_y^4}{n-1} + \dfrac{s^4}{m-1}}$$

(and we will always round **down** to an integer degrees of freedom).

One Method of Separating Process and Measurement Variation

Example 9-1

In Module 5, we considered $m = 5$ measurements made by a single analyst on a single physical sample of material using a particular assay machine that produced $s = .0120$. Suppose that subsequently, samples from $n = 20$ different batches are analyzed and $s_y = .0300$. An estimate of real process standard deviation (uninflated by measurement variation) is then

$$\widehat{\sigma_x} = \sqrt{\max\left(0, s_y^2 - s^2\right)} = \sqrt{\max\left(0, (.0300)^2 - (.0120)^2\right)} = .0275$$

and this value can used to make confidence limits. The Satterthwaite "approximate degrees of freedom" are

$$\hat{\nu} = \frac{\widehat{\sigma_x}^4}{\dfrac{s_y^4}{n-1} + \dfrac{s^4}{m-1}} = \frac{(.0275)^4}{\dfrac{(.0300)^4}{19} + \dfrac{(.0120)^4}{4}} = 11.96$$

and rounding down to $\hat{\nu} = 11$, an approximate 95% confidence interval for the real process standard deviation, σ_x, is

$$\left(.0275\sqrt{\frac{11}{21.920}}, .0275\sqrt{\frac{11}{3.816}}\right) \quad \text{i.e.} \quad (.0195, .0467)$$

A Second Application of the Statistical Method

The insight just applied to separating process and measurement variation is that of taking a difference between two sample variances, one estimating variability due to two sources (process and measurement), and the other representing variation due to only one of the two components (measurement). This thinking can also be used to separate components of measurement variation. That is, it provides a first very simple way of separating repeatability variation and reproducibility variation.

Now use the notation y for single measurements on an item made by n different operators (devices) and the notation y' for m repeat measurements made on that item by a single operator (device). A combination of things said in Modules 2 and 3 implies that the sample standard deviation of the y's, s_y, is a natural empirical approximation for $\sigma_y = \sqrt{\sigma_\delta^2 + \sigma_{\text{device}}^2}$ while the sample standard deviation of the y''s, s, is a natural empirical approximation for σ_{device}. This is pictured in the next figure.

A Second Application of the Statistical Method

$$y_i\text{'s} \sim \text{ind}\left(x + \mu_\delta, \sqrt{\sigma_\delta^2 + \sigma_{\text{device}}^2}\right)$$

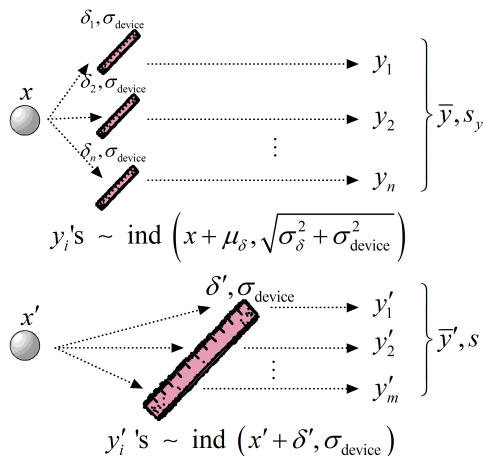$$y_i'\text{'s} \sim \text{ind}\left(x' + \delta', \sigma_{\text{device}}\right)$$

Figure: A Simple Data Collection Plan Allowing the Separation of Repeatability and Reproducibility Variances (Assuming σ_{device}, i.e. Repeatability, is Constant)

In this second context

$$\widehat{\sigma}_\delta = \sqrt{\max\left(0, s_y^2 - s^2\right)}$$

estimates the reproducibility standard deviation (the standard deviation of device biases). Then, the formulas on panel 5 with $\widehat{\sigma}_\delta$ replacing $\widehat{\sigma}_x$ provide approximate confidence limits (not for the process standard deviation but) for the reproducibility standard deviation, σ_δ.

Suppose that in an IE 361 style measurement exercise, a single student measures the size of a polystyrene packing peanut $m = 5$ times using a crude plastic caliper with a sample standard deviation of $s = .012\,\text{in}$. Suppose that subsequently, $n = 6$ different students measure a single peanut once each with a resulting standard deviation of $s_y = .030\,\text{in}$. An estimate of reproducibility standard deviation (uninflated by repeatability variation) is then

$$\widehat{\sigma}_\delta = \sqrt{\max\left(0, s_y^2 - s^2\right)} = \sqrt{\max\left(0, (.030)^2 - (.012)^2\right)} = .0275\,\text{in}$$

and this value can used to make confidence limits for σ_δ. The Satterthwaite "approximate degrees of freedom" are

$$\hat{\nu} = \frac{\widehat{\sigma}_\delta^{\,4}}{\dfrac{s_y^4}{n-1} + \dfrac{s^4}{m-1}} = \frac{(.0275)^4}{\dfrac{(.030)^4}{5} + \dfrac{(.012)^4}{4}} = 3.42$$

and rounding down to $\hat{\nu} = 3$, an approximate 95% confidence interval for the reproducibility standard deviation, σ_δ, is

$$\left(.0275\sqrt{\frac{3}{9.348}}, .0275\sqrt{\frac{3}{.216}} \right) \quad \text{i.e.} \quad (.016\,\text{in}, .103\,\text{in})$$

In contrast to this inference, notice that applying the basic confidence limits for a standard deviation (based on $s = .012$ and $\nu = m - 1 = 5 - 1 = 4$), 95% limits for the the repeatability standard deviation, σ_{device}, here are

$$\left(.012\sqrt{\frac{4}{11.143}}, .012\sqrt{\frac{4}{.484}} \right) \quad \text{i.e.} \quad (.007\,\text{in}, .034\,\text{in})$$